

Bipolaronic charge excitations in t-J two-leg ladders

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We present a low-energy effective model for the charge degrees of freedom in two-leg t-J ladders. Starting from $SU(2)$ mean-field theory, we exclude the spin degrees of freedom which have an energy gap. At low temperatures, the mean-field solution is the staggered-flux phase. For gapless charge excitations the effective theory is the Luther-Emery liquid. Our analysis is applicable at low doping and in the “physical” range of parameters $t/J \sim 3$ where there is only one massless mode in the charge sector and no massless modes in the spin sector. Within our model we make predictions about correlation exponents and the superconductivity order parameter, and discuss the comparison with the existing numerical results.

I. INTRODUCTION

Theoretical studies of t-J ladders have proven to be valuable for understanding $Hi-T_c$ cuprate compounds. While containing certain features of the two-dimensional t-J model, the ladders are quasi-one-dimensional, which greatly simplifies the treatment of the problem. Recently, cuprate compounds with ladder structures have been produced experimentally, and data on their electronic properties have been obtained [1]. The ladder compounds may be used for verification of models and mechanisms of superconductivity proposed for layered cuprates. The experimental works are now complemented by numerical results on t-J and Hubbard models [2,3]. Together with experimental results, they provide a good testing ground for any analytical treatment.

The challenge of theoretically solving the t-J model on ladders arises from its strongly correlated nature. In the real ladder compounds the coupling is nearly isotropic, i.e. the interchain coupling parameters (t and J) are close to the intrachain coupling. Thus, the problem does not have a small parameter and cannot be treated by a standard perturbation theory. Several works exist based on starting from uncoupled chains and then including interchain hopping and spin exchange as perturbations [4,5]. While weak-coupling approach is the most consistent and controlled of the existing analytic methods, it is not completely reliable as the interchain coupling increases and approaches the single-chain bandwidth. We shall further comment on possible corrections to this treatment.

In the present paper we employ the $SU(2)$ slave-boson mean-field approach [6,7]. Although not a controlled approximation, we believe that it can correctly capture the low-energy physics of the systems with a spin gap. In the paper we specialize to two-leg ladders, but our treatment may be further extended to any even-leg ladders which are known to exhibit spin gap. The spin gap ensures that most of the fluctuations around the mean-field state are massive. The only massless fluctuation is the 1+1-dimensional abelian gauge field which can be explicitly included as a pair-binding potential.

The general idea of slave-boson method is to represent the vacant sites (holes) by an auxiliary bosonic field, which allows us to rewrite the non-linear no-double-occupancy condition as a linear constraint in terms of fermions (representing spin degrees of freedom) and bosons (representing charge degrees of freedom) [6–8]. Introducing auxiliary bosons expands the Hilbert space of states, and the system acquires an additional gauge symmetry. The mean-field ansatz breaks this symmetry, which is restored for physical correlation functions after averaging over all gauge-equivalent configurations.

We use the $SU(2)$ version of the slave-boson construction developed earlier for the two-dimensional problem [6,7]. In this method, the auxiliary boson has two components (we call the corresponding degree of freedom isospin; it is distinct from the actual spin) which describe holes in the two different ways: either as sites with no fermions or as sites doubly occupied by fermions. Thus extended, the system has a $SU(2)$ gauge symmetry (rotating isospin). We choose to use the $SU(2)$ formalism instead of $U(1)$ version of the slave-boson method developed previously for two-dimensional t-J model [8,9]. We note that both analytic and numerical works [2,5,10] point to a bipolaron picture where the holes are bound in pairs. As we shall see, this picture naturally emerges out of the $SU(2)$ formulation as the confinement between two species of bosons, while the $U(1)$ formalism fails to give the correct low-energy physics. We believe that the $SU(2)$ mean-field theory has an advantage at low doping where it generalizes the $SU(2)$ symmetry of t-J model at half-filling [6]. A discussion of relation between $U(1)$ and $SU(2)$ approaches may be found in [7].

We find that the low-temperature mean-field phase is the staggered-flux phase (similar to that found for the two-dimensional problem in [6,7]). The fluctuations about this mean-field state are described by gauge fields which have a gap, except for the in-phase fluctuations which form a 1+1-dimensional abelian gauge field. As a result, we find that the low-energy effective theory consists of two degenerate bands of holons with a short-range interaction and coupled to a $U(1)$ gauge-field. The dynamics of this gauge field arises from its interactions with spinons. Spinons have a gap and, therefore, give a nonsingular dynamics to the gauge field with the energy scale J . The two bands of holons have opposite charges with respect to this gauge field and, therefore, form confined pairs. This leads us to the conclusion that the resulting theory for the hole excitations is the Luther-Emery liquid of hole pairs. This agrees with the bipolaronic picture of charge excitations proposed earlier both analytically and numerically [2,5,10]. It is known that the Luther-Emery liquid has two competing orders: superconducting singlet pairing (SS) and charge density wave (CDW) [4,5,11]. We point out the necessity to distinguish between the hole density (two-particle operator) and the pair density (four-particle operator). While the product of the correlation exponents for the SS and CDW order parameters is equal to one when CDW is understood as hole-density correlations, this relation does not necessarily hold for four-particle pair-density correlations. The relation between the single-hole and pair CDW exponents depends on the degree of the overlap of the bipolaronic pairs. At low density of holes, when the pairs do not overlap, these two exponents coincide. On the other hand, in the limit of highly overlapping pairs we find that the effective exponents may differ by 2. This possibly explains the unexpected numerical results for the correlation exponents obtained by Noack et al. [2].

Further, we discuss the possible implication of our model for the superconducting transition via pair condensation (of course, interladder correlations would be necessary). We describe the superconducting order by the nearest-neighbor order parameter Δ_{ij} . We extend our discussion for a more general case of a weakly doped antiferromagnet on a bi-partite lattice with a spin gap. This class of systems includes all even-leg ladders as a particular case. We assume that at low temperature such a system is in the staggered flux phase, which results in two degenerate interacting holonic bands. Under these assumptions we find that the order parameter obeys the modified d-wave relation:

$$\sum_j t_{ij} \Delta_{ij} = 0, \quad (1)$$

where the sum is performed over all nearest neighbors of a site i . This relation holds in the limit of zero doping, with corrections involving the hole concentration. This relation was first derived by S. C. Zhang as an exact result for the Hubbard model [12]. Our derivation should be understood as a verification that the $SU(2)$ mean-field approximation preserves this exact property. When specialized to the case of the two-leg ladder with isotropic coupling, the above equation becomes $\Delta_{\perp} = -2\Delta_{\parallel}$. This agrees very well with the earlier numerical results [2,3].

The rest of the paper is organized in three sections. In the first section, we review the $SU(2)$ slave-boson method and present the results of the mean-field theory computations. In the second section we discuss the effective theory for the holons and the correlation exponents for SS and CDW pairing. Finally, the third part is devoted to the discussion of the modified d-wave relation for the superconducting order parameter.

II. $SU(2)$ MEAN-FIELD THEORY OF THE LADDER

In this section we present the $SU(2)$ mean-field theory for the t-J Hamiltonian

$$H = \sum_{\{ij\}} J(\vec{S}_i \vec{S}_j - \frac{1}{4} n_i n_j) - t \mathcal{P}(c_{\alpha i}^{\dagger} c_{\alpha j} + \text{h.c.}) \mathcal{P} \quad (2)$$

on the ladder (Fig. 1).

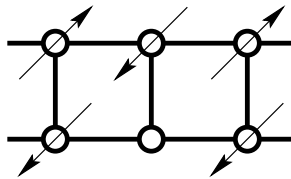


FIG. 1. Two-leg t-J ladder.

The sum is performed over nearest-neighbor site pairs $\{i, j\}$, $c_{\alpha i}^\dagger$ and $c_{\alpha i}$ are the electron creation and annihilation operators on site i (α is the spin index), \vec{S}_i is the electron spin, $\vec{S}_i = \frac{1}{2}c_{\alpha i}^\dagger \vec{\sigma}_{\alpha\beta} c_{\beta i}$, n_i is the occupation number of the site i ($n_i = c_{\alpha i}^\dagger c_{\alpha i}$), \mathcal{P} is the projector onto the no-double-occupancy states (with $n_i \leq 1$ for any i). t and J are the parameters of the Hamiltonian. In the real ladder compounds t and J are estimated to be about 4000K and 1300K respectively [1]. In what follows we assume that the interchain and the intrachain couplings are equal ($t_\perp = t_\parallel = t$, $J_\perp = J_\parallel = J$) and as a realistic approximation we take $t/J = 3$.

Following the usual procedure of the $SU(2)$ slave-boson method [6,7], we introduce two fermionic and one bosonic isospin doublets on each site:

$$\psi_{1i} = \begin{pmatrix} f_{1i} \\ f_{2i}^\dagger \end{pmatrix} \quad \psi_{2i} = \begin{pmatrix} f_{2i} \\ -f_{1i}^\dagger \end{pmatrix} \quad h_i = \begin{pmatrix} b_{1i} \\ b_{2i} \end{pmatrix} \quad (3)$$

with the electronic operators written in terms of bosons h_i and fermions $\psi_{\alpha i}$ as

$$c_{\alpha i} = \frac{1}{\sqrt{2}} h_i^\dagger \psi_{\alpha i} \quad (4)$$

The resulting Hilbert space is larger than that of the original t-J system. To select the subspace of physical states (which is invariant under the Hamiltonian of the original system) we impose a linear constraint (replacing the non-linear no-double-occupancy constraint):

$$\left(\frac{1}{2} \psi_{\alpha i}^\dagger \vec{\tau} \psi_{\alpha i} + h_i^\dagger \vec{\tau} h_i \right) |phys\rangle = 0 \quad (5)$$

($\vec{\tau}$ are identical to the Pauli matrices $\vec{\sigma}$, but they act in the isospin space, and we denote them by a different letter to distinguish from the Pauli matrices $\vec{\sigma}$ acting on true spin). On a given site, this constraint allows only three states: $f_1^\dagger|0\rangle$, $f_2^\dagger|0\rangle$, and $\frac{1}{\sqrt{2}}(b_1^\dagger + b_2^\dagger f_1^\dagger f_2^\dagger)|0\rangle$, which correspond to spin up, spin down electrons and a vacancy respectively. Thus in the $SU(2)$ formulation, a vacancy may be represented by both two-spinon and no-spinon states corresponding to different isospins of the holon h_i .

Thus formulated, the extended system is invariant under an $SU(2)$ gauge symmetry:

$$\psi_{\alpha i} \mapsto g_i \psi_{\alpha i}, \quad h_i \mapsto g_i h_i. \quad (6)$$

This gauge symmetry acts on the isospin of fermions and bosons, and mixes creation operators $f_{\alpha i}^\dagger$ with the annihilation operators of opposite spin $f_{-\alpha i}$.

Introducing the nearest-neighbor mean-field parameters

$$U_{ij} = \langle \psi_{\alpha i} \psi_{\alpha j}^\dagger \rangle - \frac{t}{J} (\langle h_i h_j^\dagger \rangle + \mathbf{I} \langle h_j h_i^\dagger \rangle^T \mathbf{I}) \quad (7)$$

with

$$\mathbf{I} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (8)$$

and the Lagrange multipliers a_i^μ ($\mu = 1, 2, 3$) for enforcing the linear constraint (5), the mean-field Hamiltonian becomes

$$H = \sum_{\{ij\}} \left[\frac{J}{4} \text{Tr}(U_{ij} U_{ij}^\dagger) + \frac{J}{2} \psi_{\alpha i}^\dagger U_{ij} \psi_{\alpha j} + \frac{t}{2} (h_i^\dagger U_{ij} h_j + \text{h.c.}) \right] + \sum_i a_i^\mu \left(\frac{1}{2} \psi_{\alpha i}^\dagger \tau_\mu \psi_{\alpha i} + h_i^\dagger \tau_\mu h_i \right). \quad (9)$$

In the present paper we use combinatoric coefficients in (9) different from those used in [6,7,9]. Our choice of coefficients in (9) gives the correct combinatoric factors for tadpole diagrams, and they differ from those obtained from the Hubbard-Stratonovich transformation. This ambiguity in the numeric factors of order one is of no practical significance, since it falls within the uncertainty of the mean-field approximation.

The mean-field Hamiltonian (9) is accurate only up to four-boson terms which we neglect in the usual way [8]. The four-boson terms give rise to only short-range interactions between holons and can be omitted at this level of approximation.

In [6] it was argued that the non-zero values of a_i^μ correspond to Bose condensation. We do not expect Bose condensation in a quasi-one-dimensional system and set mean-field value $a_i^\mu = 0$. This implies that we in fact release

the no-double-occupancy constraint (it is satisfied only on average). Afterwards, the constraint may be imposed by including the fluctuations of the field a_i^μ .

Introducing the chemical potential μ , we arrive at the mean-field Hamiltonian

$$H = \sum_{\{ij\}} \left[\frac{J}{4} \text{Tr}(U_{ij} U_{ij}^\dagger) + \frac{J}{2} \psi_{\alpha i}^\dagger U_{ij} \psi_{\alpha j} + \frac{t}{2} (h_i^\dagger U_{ij} h_j + \text{h.c.}) \right] - \mu \sum_i (h_i^\dagger h_i - \delta), \quad (10)$$

where δ is the concentration of holes (doping). The matrices U_{ij} have the form

$$U_{ij} = \begin{pmatrix} \chi_{ij} & \Delta_{ij} \\ \Delta_{ij}^* & -\chi_{ij}^* \end{pmatrix} = U_{ji}^\dagger = i a_{ij} G_{ij}, \quad (11)$$

where a_{ij} are positive real numbers (amplitudes), $G_{ij} \in SU(2)$ are 2×2 matrices.

The mean-field saddle point at temperature T is found as the extremum of the free energy

$$F[U_{ij}] = -T \log \text{Tr}_{h,\psi} \exp(-H/T). \quad (12)$$

$F[U_{ij}]$ is invariant under $SU(2)$ gauge transformations

$$U_{ij} \mapsto W_i U_{ij} W_j^\dagger \quad (13)$$

for any set of $SU(2)$ elements W_i .

The mean-field solution breaks this gauge symmetry. Only gauge invariant quantities correspond to physical observables. Any non-gauge-invariant expression will vanish after averaging over all gauge-equivalent configurations of U_{ij} .

Now we turn to describing possible phases. Phases should be parametrized by gauge-invariant functions of U_{ij} . We assume that the translational symmetry is unbroken, i.e. a translation of the mean-field solution $\{U_{ij}\}$ along the ladder transforms it to a gauge equivalent configuration. We also assume that the symmetry of reflection about the ladder axis (interchanging the two legs) is also preserved in the mean-field solution. Under these assumptions all possible phases may be parametrized by four real parameters: the two amplitudes a_{\parallel} and a_{\perp} (intrachain and interchain respectively), and two $SU(2)$ order parameters:

$$b = \frac{1}{2} \text{Tr} \prod_{\Gamma_1} G_{ij}, \quad (14)$$

$$c = \frac{1}{2} \text{Tr} \prod_{\Gamma_2} G_{ij} \quad (15)$$

with the products taken along the contours Γ_1 and Γ_2 shown in Fig. 2 (the first product contains four matrices, the second one — eight matrices).

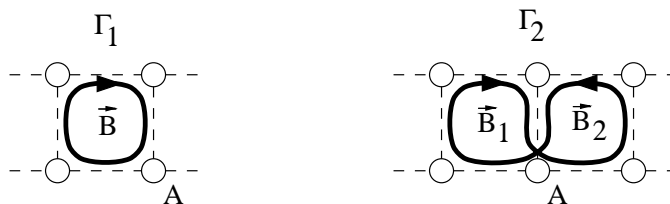


FIG. 2. The two closed contours used in constructing mean-field order parameters.

The meaning of the order parameter b is analogous to the cosine of the flux through plaquet in the $U(1)$ formulation [8]. To explain this analogy we may introduce the $SU(2)$ flux \vec{B} defined by

$$\exp(i\vec{B} \cdot \vec{\tau}) = \prod_{\Gamma_1} G_{ij}, \quad (16)$$

where the product starts and ends at a site A of the contour Γ_1 (Fig. 2). Then $b = \cos |\vec{B}|$ (obviously, the direction of the vector \vec{B} depends on the choice of the starting point A , but its magnitude $|\vec{B}|$ does not).

The order parameter c measures the relative orientation of neighboring $SU(2)$ fluxes. Namely, define the two fluxes \vec{B}_1 and \vec{B}_2 through neighboring plaquets with a common starting point A (Fig. 2). Then

$$c = \frac{1}{2} \text{Tr} \left(\exp(i\vec{B}_1 \cdot \vec{\tau}) \exp(-i\vec{B}_2 \cdot \vec{\tau}) \right) = \cos |\vec{B}_1| \cos |\vec{B}_2| + \frac{(\vec{B}_1 \cdot \vec{B}_2)}{|\vec{B}_1| |\vec{B}_2|} \sin |\vec{B}_1| \sin |\vec{B}_2|. \quad (17)$$

Then we see that for any $G_{ij} \in SU(2)$ the order parameters b and c are restricted to

$$-1 \leq b \leq 1, \quad (18)$$

$$2b^2 - 1 \leq c \leq 1. \quad (19)$$

Thus, the $SU(2)$ order may be represented by a point in a two-dimensional domain (Fig. 3) with the two corners representing the π -flux and the uRVB phases, the boundaries corresponding to the uniform flux (uF) and staggered flux (sF) phases analogous to their $U(1)$ counterparts [9], but preserving translational and time-reversal symmetries. In the uF phase the neighboring $SU(2)$ fluxes are parallel, in the sF phase they are antiparallel, and inside the shaded region in Fig. 3 they form angles ranging between 0 and π .

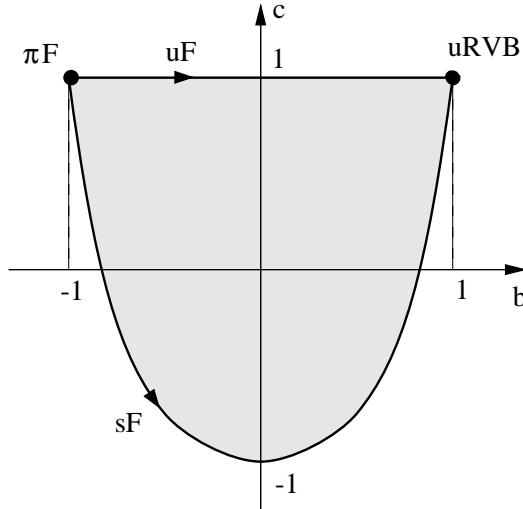


FIG. 3. Space of mean-field phases. The two corners correspond to the π -flux and uRVB phases, boundaries — to the uniform-flux and staggered-flux phases.

By numerically minimizing the free energy (12) (at $t/J = 3$) we find the following mean-field phase diagram in the (δ, T) coordinates:

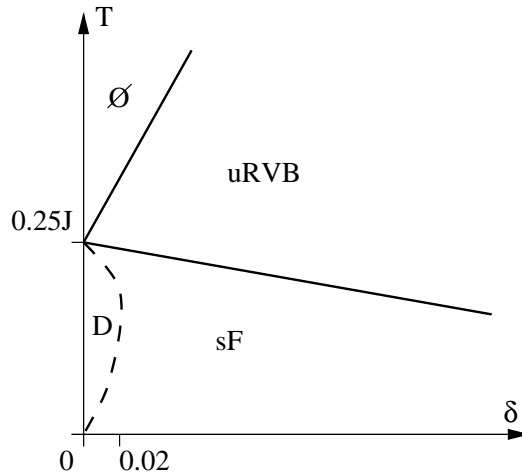


FIG. 4. Mean-field phase diagram at isotropic coupling and $t/J = 3$.

where

1. \emptyset denotes the high-temperature free spin phase ($a_{\parallel} = a_{\perp} = 0$);
2. uRVB is the phase with $a_{\parallel} \neq 0$, $a_{\perp} \neq 0$, $G_{ij} = \mathbf{1}$ (so that $b = c = 1$);
3. D is the dimer phase: $a_{\perp} \neq 0$, $a_{\parallel} = 0$;
4. sF is the staggered-flux phase with $a_{\perp} \neq 0$, $a_{\parallel} \neq 0$, $-1 < b < 1$, $c = 2b^2 - 1$.

Numerically we find that the transition between D and sF phases is a very soft first-order transition. In fact, the dimer phase D has flat spectra for bosons and fermions and will be destroyed by the fluctuations (correlations along the ladder will appear). We shall disregard the dimer phase as an artifact of the mean-field approximation and for the rest of the paper we restrict our discussion to the sF phase.

The sF phase may be described by different gauge-equivalent configurations of U_{ij} . One of the *translationally invariant* configurations (analogous to d-wave pairing phase in the $U(1)$ mean-field theory [9]) is (Fig. 5):

$$G_{\perp} = i\tau_1, \quad G_{\parallel} = i(\cos \frac{\varphi}{2} \tau_1 + \sin \frac{\varphi}{2} \tau_2), \quad (20)$$

or, equivalently,

$$U_{\perp} = \begin{pmatrix} 0 & a_{\perp} \\ a_{\perp} & 0 \end{pmatrix}, \quad U_{\parallel} = \begin{pmatrix} 0 & a_{\parallel} e^{i\frac{\varphi}{2}} \\ a_{\parallel} e^{-i\frac{\varphi}{2}} & 0 \end{pmatrix}. \quad (21)$$

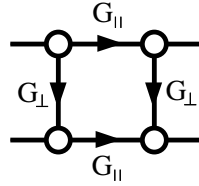


FIG. 5. sF phase order parameters in the translationally invariant gauge.

Further, we shall use a different, the so called “abelian” parametrization with

$$G_{\parallel} = \mathbf{1}, \quad G_{\perp} = \cos \frac{\varphi}{2} + (-1)^m i \tau_3 \sin \frac{\varphi}{2}, \quad (22)$$

(Fig. 6) which corresponds to

$$U_{\parallel} = i \begin{pmatrix} a_{\parallel} & 0 \\ 0 & a_{\parallel} \end{pmatrix}, \quad U_{\perp} = i \begin{pmatrix} a_{\perp} e^{i(-1)^m \frac{\varphi}{2}} & 0 \\ 0 & a_{\perp} e^{-i(-1)^m \frac{\varphi}{2}} \end{pmatrix}, \quad (23)$$

where m is the number of the rung. This gauge fixing is not translationally invariant, but instead it has the property that all U_{ij} commute. This choice of gauge resembles the staggered-flux $U(1)$ phase [9]. In the $U(1)$ formalism, the staggered-flux and d-wave pairing phases are different, but their $SU(2)$ counterparts are gauge-equivalent [6].

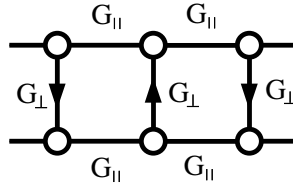


FIG. 6. sF order parameters in the abelian gauge.

In the $SU(2)$ mean-field theory bosonic and fermionic spectra are proportional to each other (with the scales t and J respectively). In the sF phase we obtain the spectrum

$$\frac{E_f}{J} = \frac{E_b}{t} = \pm \frac{1}{2} \sqrt{a_{\perp}^2 + (2a_{\parallel})^2 \cos^2 k + 2(2a_{\parallel})a_{\perp} \cos \frac{\varphi}{2} \cos k} \quad (24)$$

($k \in [0; 2\pi]$ is the wave vector) and each of these two bands is doubly degenerate (four bands total). We remark that labeling the states by wave vectors k may depend on the gauge, but, because gauge-dependent shift of k involves equally bosons and fermions, the gauge-invariant quantities remain unchanged.

The double degeneracy of states is a characteristic feature of the staggered-flux phase. It is due to the fact that a certain non-abelian subgroup of the full symmetry group of the extended Hamiltonian remains unbroken by the mean-field ansatz. In the staggered-flux phase the order parameters U_{ij} are invariant under a $U(1)$ subgroup of global isospin rotations (in the abelian gauge (22) – (23) these are simply global rotations by τ_3). Besides, there remains a particular symmetry of U_{ij} , which is the combination of time reversal (transforming $U_{ij} \mapsto U_{ij}^*$) and global isospin rotation (exchanging isospins up and down in the abelian gauge). This symmetry operation does not commute with the $U(1)$ rotation, but extends them to a nonabelian group. The two degenerate bands form a two-dimensional representation of this group, with time reversal mapping one band onto the other.

The typical numerical values for a_{\parallel} , a_{\perp} and $\cos \frac{\varphi}{2}$ are $a_{\parallel} = 0.5$, $a_{\perp} = 0.9$, $\cos \frac{\varphi}{2} = 0.4$ (found by minimizing free energy at $\delta = 0.05$, $T = 0.1J$, $t/J = 3$). This means that the upper bands of the spectrum are separated from the lower bands by a gap of order J for fermions and of order t for bosons (Fig. 7). The fermionic spectrum is half-filled, i.e. the lower bands are completely filled whereas the upper bands are empty. The fermionic excitations have a gap of order of J . This agrees with the prediction of spin gap in two-leg ladders and is crucial for the stability of the phase.

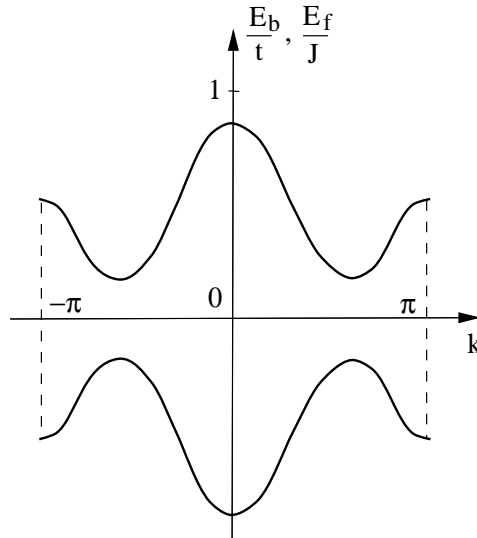


FIG. 7. Typical fermionic-bosonic spectrum. Both bands are doubly degenerate.

Finally, we need to include the fluctuations about the mean-field phase. These fluctuations are described by the spatial $SU(2)$ gauge fields $A_{ij} = U_{ij}^{-1} \delta U_{ij}$ defined on the links and their temporal counterparts $a_i = a_i^\mu \tau_\mu$ defined on the sites [6–8]. The temporal components of the gauge field a_i coincide with the constraint-fixing Lagrange multipliers in (9) when expressed in units of J . The effective action for the fields A_{ij} and a_i is invariant under the (time-dependent) gauge transformations

$$A_{ij} \mapsto W_i(t) A_{ij} W_j^\dagger(t), \quad a_i \mapsto a_i + i \partial_t W_i(t) \quad (25)$$

for arbitrary $SU(2)$ matrices $W_i(t)$ defined on sites.

In the sF phase the $SU(2)$ gauge symmetry is broken down to $U(1)$ global symmetry. In the abelian gauge, this residual symmetry is realized by rotations by τ_3 . As shown in the Appendix B of [7], the free energy will contain terms proportional to $\text{Tr} \prod U_{ij}$, where the products are taken along closed loops on the lattice. When expanded in gauge-field fluctuations, these terms give rise to mass for the gauge modes proportional to τ_1 and τ_2 . Thus, massless modes of the gauge field must be proportional to τ_3 [7], and we may treat A_{ij} and a_i as $U(1)$ gauge fields.

To proceed further, we may make use of the symmetry (25) (now an $U(1)$ gauge symmetry with all $W_i(t)$ being rotations by τ_3). The analysis appears to be particularly simple in the gauge where $A_{ij} = 0$ across the rungs (Fig. 8).

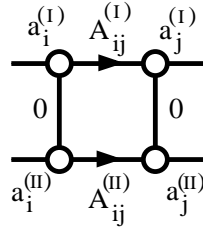


FIG. 8. Gauge fields describing the fluctuations of the order parameter. We choose the gauge with $A_{\perp} = 0$.

Then the remaining gauge fields split into the in-phase and out-of-phase modes

$$A_{\pm} = A_{ij}^{(I)} \pm A_{ij}^{(II)}, \quad a_{\pm} = a_i^{(I)} \pm a_i^{(II)} \quad (26)$$

(where the superscripts (I) and (II) label the two legs of the ladder). The dynamics of the gauge fields arises from their coupling to the spinons and can be found by computing the polarization diagram (Fig. 9), solid lines denote spinons):

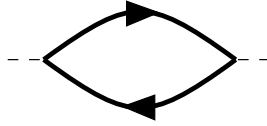


FIG. 9. Lowest-order diagram responsible for the dynamics of the gauge field. Solid lines denote spinon propagators. This diagram gives mass to the out-of phase mode and 1+1-dimensional QED dynamics for the in-phase mode.

In contrast to the two-dimensional model, where there exist massless transverse fluctuations of the gauge field, it is not the case in the ladder. The ladder geometry restricts the transverse wavevector to two values $k_{\perp} = 0$ and $k_{\perp} = \pi$ which correspond to the modes (A_+, a_+) and (A_-, a_-) respectively. The out-of-phase modes A_- and a_- (describing the fluctuations of the flux φ through the plaquet) acquire a finite mass of order J and, therefore, can be neglected, giving only a short-range interactions between holons.

On the other hand, the modes A_+ and a_+ become 1+1 QED gauge fields with the long-wavelength action

$$S = \frac{1}{J^*} \int (\partial_t A_+ + \partial_x a_+)^2 dx dt \quad (27)$$

with J^* of order J . This action arises from expanding the polarization diagram in Fig. 9. The particular value of J^* depends on the values of mean-field parameters, and for the typical values cited above differs from J only by a factor of order unity.

Since the two lower bosonic bands have different isospin, they have opposite charges with respect to the gauge field (A_+, a_+) . In 1+1 dimension, electromagnetic field leads to a confining (linearly growing with the distance) potential between charges:

$$U(r) = \pm J^* |r|. \quad (28)$$

In the limit of low hole density, the bosons will form isospin-neutral dipole pairs (bipolarons) which produce no field outside each pair. Therefore, bipolarons will interact only by short-range forces.

We may estimate the size of bipolaron by solving a simple quantum-mechanical problem of two particles interacting via the potential (28). The holon hopping amplitude is of order t , therefore the kinetic energy of bipolaron is of order $t\xi_{pair}^{-2}$. The potential energy, on the other hand, is of order $J\xi_{pair}$. Thus from variational principle we find that the size of bipolaron is $\xi_{pair} \sim (t/J)^{1/3}$ up to a factor of order one. For our assumption $t/J = 3$, this gives $\xi_{pair} \sim 1$, which means that now the other short-range (repulsive) terms, which we omitted before, give a comparable contribution. Due to the no-double-occupancy constraint, the two species of bosons must be subject to a substantial on-site repulsion. It increases the estimated size of the pair by several lattice spacings, but does not change the long-range attractive force. Since we omitted the short-range part of the interaction from the very beginning, we

cannot compute the size of the pair more precisely. From the numerics [2] we know that the characteristic decay length for the pairing correlation function is about four lattice spacings, in agreement with our discussion.

A simple classical explanation of the confinement may be obtained from the picture of fluctuating singlet bonds (somewhat in the spirit of [10]). We may think of the spin structure of the Heisenberg antiferromagnet on the ladder as of spins forming fluctuating singlet bonds with nearest neighbors (Fig. 10).

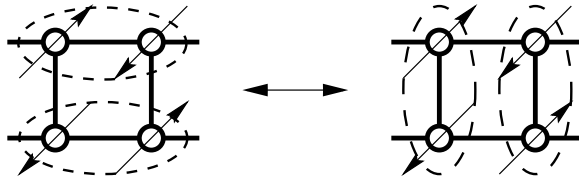


FIG. 10. Fluctuating singlets in the spin ladder.

Once we have a single hole, it leads to the appearance of a localized spin, which costs a finite energy of order J . When putting two holes, the spins between the holes must form singlet bonds in a non-favorable way without a freedom to fluctuate (Fig. 11).

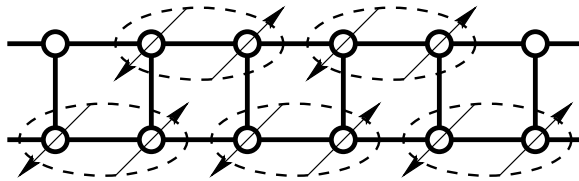


FIG. 11. Misplaced singlet bonds between two holes in the t-J ladder. The energy cost is proportional to the distance between the holes.

Naturally, this string of singlets costs certain energy J^*r , where r is the distance between the holes and J^* is of order J . We believe that this naive picture gives a correct understanding of the holon confinement which we derived starting from the mean-field sF phase.

III. LUTHER-EMERY LIQUID AND CORRELATION EXPONENTS

In the previous section we have shown that the low-energy excitations in our model are pairs of holons — bipolarons — which are bound by long-range confining interaction. Due to complete screening of the confining interaction by a single particle in one dimension, bipolarons interact only via short-range forces, and this recovers the picture of Luther-Emery liquid of hole pairs proposed earlier in [10].

In fact, this simple picture is valid at low hole concentration $\delta \ll \xi_{pair}^{-1}$, when bipolarons do not overlap. The purpose of this section is to show that as overlap increases, the charge-density-wave exponents for single-particle and for pair densities may differ.

For quantitative description of bipolarons we introduce the three different correlation exponents as follows. Let $n(x) = n^{(I)} + n^{(II)} - 2\delta$ be the fluctuation of the number of holes on the rung (x is the coordinate along the ladder, superscripts refer to the two chains). Let further $n_{pair} = (n^{(I)} - \delta)(n^{(II)} - \delta)$ be the fluctuation of the probability that both sites of the same rung contain holes. Finally, let $\Delta(x) = c_{\downarrow}^{(I)} c_{\uparrow}^{(II)} - c_{\uparrow}^{(I)} c_{\downarrow}^{(II)}$ be the singlet superconducting order parameter on the rung at the position x . Then define the correlation exponents α_1 , α_2 and γ by

$$\langle n(x)n(y) \rangle_{2k_F} \propto |x - y|^{-\alpha_1}, \quad (29)$$

$$\langle n_{pair}(x)n_{pair}(y) \rangle_{2k_F} \propto |x - y|^{-\alpha_2}, \quad (30)$$

$$\langle \Delta^\dagger(x)\Delta(y) \rangle \propto |x - y|^{-\gamma}, \quad (31)$$

where $\langle \dots \rangle_{2k_F}$ is the coefficient at $\cos[2k_F(x - y)]$ in the expansion of the correlation function [14]:

$$\begin{aligned} \langle n(x)n(y) \rangle = & A_0|x - y|^{-\theta_0} + A_1|x - y|^{-\theta_1} \cos[2k_F(x - y)] + \\ & + A_2|x - y|^{-\theta_2} \cos[4k_F(x - y)] + \dots \end{aligned} \quad (32)$$

The correlation function (29) describes charge-density-wave (CDW) ordering, and the correlation function (31) — singlet superconductivity (SS) ordering.

By k_F we denote the “Fermi wavevector” for the holons in the lower bands, so that $2k_F = 2\pi\delta$. For comparison with other works, we must remark that $2k_F$ in our notation corresponds to $4k_F$ in the notation of [2] and [5].

In the limit of dilute gas of bipolarons (the doping is much less than the inverse size of a bipolaron: $\delta \ll \xi_{pair}^{-1}$) we may describe the low-energy states of the system in terms of bipolaron creation and annihilation operators (similarly to the large $|U|$ limit in the attractive Hubbard model [11,13]). Bipolarons interact repulsively and, from a naive classical picture, it is likely that the repulsion is nearly hard-core. In this limit $\alpha_1 = \alpha_2$, because on the bipolaronic subspace of the total Hilbert space the matrix elements of the operators $n(x)$ and $n_{pair}(x)$ differ only by a numerical factor (the probability of the two holons in the pair to occupy the same rung). For hard-core repulsion $\alpha_{1,2} = 2$, since hard-core bosons can be mapped to free fermions by a Jordan-Wigner-type transformation, so that density-density correlations coincide with those of free fermions. From the theory of Luther-Emery liquid it is known that CDW and SS correlations are described by dual phases, and the corresponding exponents are therefore reciprocal: $\gamma = 1/\alpha$ [11,13]. A more detailed discussion of the Luther-Emery theory of bipolaronic excitations may be found in [10].

The opposite limit of highly overlapping bipolarons is more subtle. In this limit ($\delta \ll (t/J)^{1/3}$) bipolarons can exchange particles. Exchanging a particle would cost interaction energy of order $J\delta^{-1}$, while the gain of kinetic energy would be of order $t\delta^2$. Thus, in this limit we cannot speak of isolated bipolarons, but rather of two species of bosons with attraction much smaller than the bandwidth. We suggest that in this case we may replace the screened long-range interaction by a short-range one. In the long-wavelength limit the system may be described as two Luther-Emery liquids with a weak attraction:

$$H = H_1 + H_2 + H_{int}, \quad (33)$$

$$H_i = \frac{1}{2\pi} \int dx [v_J(\nabla\varphi_i)^2 + v_N(\nabla\theta_i)^2], \quad i = 1, 2, \quad (34)$$

$$H_{int} = -V \int dx \rho_1(x) \rho_2(x), \quad (35)$$

where φ_i and θ_i are dual phases,

$$[\varphi_i(x), \theta_j(y)] = \delta_{ij} i \frac{\pi}{2} \text{sign}(x - y), \quad (36)$$

v_J and v_N are the parameters depending on the short-range properties of the interaction, and $\rho_i(x)$ are the density fluctuations expressed by [14]

$$\rho_i(x) = \frac{1}{\pi} \nabla\theta_i(x) + 2\rho_0 \cos(2k_F x + 2\theta_i(x)) + \text{higher order terms}. \quad (37)$$

The interaction (35) contains a term proportional to $\cos 2(\theta_1(x) - \theta_2(x))$ and locks the relative phase $\theta_- = \theta_1 - \theta_2$. The only remaining gapless mode is the in-phase fluctuations ($\theta_+ = \theta_1 + \theta_2$ and $\varphi_+ = \varphi_1 + \varphi_2$) corresponding to propagation of bipolarons.

If we at first approximation neglect higher-order terms in the density expansion (37), we find

$$\langle \Delta^\dagger(x) \Delta(y) \rangle \sim \langle e^{i[\varphi_+(x) - \varphi_+(y)]} \rangle, \quad (38)$$

$$\langle n(x) n(y) \rangle_{2k_F} \sim \langle e^{i[\theta_+(x) - \theta_+(y)]} \rangle, \quad (39)$$

$$\langle n_{pair}(x) n_{pair}(y) \rangle_{2k_F} \sim \langle \nabla\theta_+(x) \nabla\theta_+(y) e^{i[\theta_+(x) - \theta_+(y)]} \rangle, \quad (40)$$

which yields $\alpha_2 = \alpha_1 + 2$, $\gamma = 1/\alpha_1$. This would explain the numerical results of [2] who found $\alpha_2 \approx \gamma \approx 2$ (at $\delta = 1/8$, $\xi_{pair} \sim 4$). Our prediction of power-law correlations (29) also explains the small, but relatively narrow peak at $2k_F$ in the Fourier transform of $\langle n(x) n(y) \rangle$ in [2].

As it was pointed out by Haldane [14], in general one must also include higher-order terms in the density expansion (37). These terms proportional to $\cos[m(2k_F x + 2\theta_i(x))]$ with $m > 1$ are absent in the free fermion theory (and,

consequently, in hard-core boson theory), but arise as we include interactions mixing left- and right-moving excitations. They produce terms proportional to $\exp i[m\theta_1(x) - (m \pm 1)\theta_2(x)]$ in the $n_{pair}(x)_{2k_F}$ expansion, and give a contribution to $\langle n_{pair}(x)n_{pair}(y) \rangle_{2k_F}$ decaying with the exponent α_1 instead of $\alpha_1 + 2$. This effect that originally higher-order terms result in leading correlation exponents is not paradoxical in view of the crossover to the dilute limit where the binding interaction is strong and the higher-order corrections to (37) play a dominating role. We suggest that the crossover from $\alpha_2 = \alpha_1$ in the dilute limit to $\alpha_2 = \alpha_1 + 2$ in the weak-coupling limit is governed by the overlap of bipolarons. Namely,

$$\langle n_{pair}(x)n_{pair}(y) \rangle_{2k_F} = \frac{A}{(x-y)^{\alpha_1}} + \frac{B}{(x-y)^{\alpha_1+2}} \quad (41)$$

with relative weights of A and B depending on the average pair overlap ($A \gg B$ at $\xi_{pair}\delta \ll 1$ and $A \ll B$ at $\xi_{pair}\delta \gg 1$). The actual behavior of the coefficients A and B strongly depends on the short-scale features of the interaction, and cannot be found in our rude treatment.

The whole discussion of this section is equally applicable to the negative- U (attractive) Hubbard model. In the low-density (or large $|U|$) limit the exponents α_1 and α_2 coincide, while in the small U limit we expect a crossover (41) to $\alpha_2 = \alpha_1 + 2$. In other words, because of screening, the long-range gauge interaction between holons in one dimension leads to the same behavior at large distances as a short-range attraction.

Finally, we comment on our disagreement with the prediction of Nagaosa [5] that the correlations $\langle n_{pair}(x)n_{pair}(y) \rangle_{2k_F}$ decay exponentially (note again that $2k_F$ in our notation corresponds to $4k_F$ in the notation of [5]). The disagreement may be explained from the fact that Nagaosa starts from two uncoupled chains and treats the interchain couplings (t_\perp and J_\perp) as perturbations. In that picture, power-law correlations $\langle n_{pair}(x)n_{pair}(y) \rangle_{2k_F}$ will appear as a correction for the nonlinearity of the spectrum as the coupling increases and approaches the bandwidth. In contrast to the weak-coupling approach, our model starts directly from diagonalizing a strong-coupling Hamiltonian (10), and the correlations (29) are present from the very beginning.

IV. MODIFIED D-WAVE RELATION ON SUPERCONDUCTING ORDER PARAMETER

In this section we verify that our approximation scheme is consistent with the exact relation for the superconducting order parameter derived by S. C. Zhang for the Hubbard model [12] (and later translated to t-J model in [15]).

Let us define the pairing operator

$$\Delta_{ij} = c_{i\downarrow}c_{j\uparrow} - c_{i\uparrow}c_{j\downarrow} \quad (42)$$

and consider the quantity

$$\Delta_{ij}^{(0)} = \langle 2|\Delta_{ij}|0 \rangle, \quad (43)$$

where $|0\rangle$ and $|2\rangle$ are the ground states with zero and two holes respectively. Should a superconducting transition happen, $\Delta_{ij}^{(0)}$ will become the superconducting order parameter.

From now on, we restrict i and j to be nearest-neighbor sites. Zhang's results states that on a bipartite lattice in the limit of zero doping $\Delta_{ij}^{(0)}$ obey the relation

$$\sum_j \Delta_{ij}^{(0)} = 0, \quad (44)$$

where the sum is over the nearest neighbors of the site i . On the two-dimensional square lattice this implies the d-wave symmetry of pairing; thus we may call Eq.(44) the modified d-wave relation.

Below we rederive this result within the $SU(2)$ slave-boson mean-field approximation. For the sake of generality, we extend our further discussion to the t-J model on an arbitrary quasi-one-dimensional bi-partite lattice, provided it exhibits a spin gap (the most popular examples of this type are even-leg ladders). Further, assume that the low-temperature mean-field phase is analogous to the sF phase of the two-leg ladder. Namely, we require that the $SU(2)$ order parameter may be brought to the diagonal form (by a suitable choice of gauge):

$$U_{ij} = \begin{pmatrix} \chi_{ij} & 0 \\ 0 & -\chi_{ij}^* \end{pmatrix}. \quad (45)$$

This requirement means that spinons and holons form doubly degenerate bands related by the symmetry of simultaneous time-reversal and isospin flip. In such a phase a superconductivity may evolve by pair formation between the holons at the bottom of the two lowest bands (of course, superconductivity is possible only when stabilized by inter-ladder interactions, see e.g. [11]).

For simplicity, let the coupling be isotropic (t and J are the same on all links), as it was assumed in the previous sections. At the end of this section we shall extend the result to non-isotropic coupling.

Using our slave-boson representation, we express Δ_{ij} (for nearest-neighbor i and j) in terms of spinons and holons as

$$\Delta_{ij} = \frac{1}{2} \left[(h_i^\dagger \psi_{2i})(h_j^\dagger \psi_{1j}) - (h_i^\dagger \psi_{1i})(h_j^\dagger \psi_{2j}) \right]. \quad (46)$$

At low doping, the fermionic part of the correlation function (4.2) may be replaced by the mean-field order parameters

$$\chi_{ij} = \langle f_{1i} f_{1j}^\dagger + f_{2i} f_{2j}^\dagger \rangle, \quad (47)$$

and we find

$$\Delta_{ij}^{(0)} = \frac{1}{2} \left(\chi_{ij} \langle 2 | b_{1i}^\dagger b_{2j}^\dagger | 0 \rangle + \chi_{ij}^* \langle 2 | b_{2i}^\dagger b_{1j}^\dagger | 0 \rangle \right), \quad (48)$$

where $|0\rangle$ and $|2\rangle$ now denote the states in the holonic sector. Let $b_1(k)$ and $b_2(k)$ be the operators destroying holons at a wave vector k in the two lowest bands (subscripts denote the isospin). Then the single-pair wave-function $|2\rangle$ has the form

$$|2\rangle = \int \frac{dk}{2\pi} \Psi_0(k) b_1^\dagger(k) b_2^\dagger(-k) |0\rangle, \quad (49)$$

where $\Psi_0(k)$ is the relative wave function of the two holons in a pair. Eq.(48) becomes

$$\Delta_{ij}^{(0)} = \int \frac{dk}{2\pi} \Psi_0(k) \Delta_{ij}(k), \quad (50)$$

where

$$\begin{aligned} \Delta_{ij}(k) &= \frac{1}{2} \left(\chi_{ij} \langle 0 | b_2(-k) b_1(k) b_{1i}^\dagger b_{2j}^\dagger | 0 \rangle + \chi_{ij}^* \langle 0 | b_2(-k) b_1(k) b_{2i}^\dagger b_{1j}^\dagger | 0 \rangle \right) = \\ &= \frac{1}{2} \left(\chi_{ij} \langle 0 | b_2(-k) b_{2j}^\dagger | 0 \rangle \langle 0 | b_1(k) b_{1i}^\dagger | 0 \rangle + \chi_{ij}^* \langle 0 | b_2(-k) b_{2i}^\dagger | 0 \rangle \langle 0 | b_1(k) b_{1j}^\dagger | 0 \rangle \right). \end{aligned} \quad (51)$$

Since $b_1(k)$ and $b_2(k)$ are related by the time-reversal symmetry (accompanied by a gauge transformation), $\langle 0 | b_2(-k) b_{2i}^\dagger | 0 \rangle = (-1)^i \langle 0 | b_{1i} b_1^\dagger(k) | 0 \rangle$, and for nearest-neighbor sites i and j

$$\begin{aligned} \Delta_{ij}(k) &= \frac{1}{2} \left((-1)^j \chi_{ij} \langle 0 | b_{1j} b_1^\dagger(k) | 0 \rangle \langle 0 | b_1(k) b_{1i}^\dagger | 0 \rangle + (-1)^i \chi_{ij}^* \langle 0 | b_{1i} b_1^\dagger(k) | 0 \rangle \langle 0 | b_1(k) b_{1j}^\dagger | 0 \rangle \right) \\ &= (-1)^i \langle k | \frac{1}{2} (-\chi_{ij} b_{1i}^\dagger b_{1j} + \chi_{ij}^* b_{1j}^\dagger b_{1i}) | k \rangle, \end{aligned} \quad (52)$$

where $|k\rangle$ is a single-holon plane wave created by $b_1^\dagger(k)$. The state $|k\rangle$ is an eigenvector of the free Hamiltonian proportional to the bosonic part of Eq.(10)

$$H_0 = \frac{1}{2} \sum_{\{ij\}} (b_{1i}^\dagger \chi_{ij} b_{1j} + b_{1j}^\dagger \chi_{ij}^* b_{1i}) \quad (53)$$

(with the sum performed over nearest-neighbor pairs of sites). Therefore

$$\sum_j \Delta_{ij}(k) = (-1)^i \langle k | [b_{1i}^\dagger b_{1i}, H_0] | k \rangle = 0, \quad (54)$$

where the sum is over the nearest neighbors of the site i . This immediately implies the result (44).

The above derivation holds also at a non-zero temperature (with ground-state expectation values replaced by thermal averages). However, it is strictly limited to zero doping: at finite doping Eq.(47) is no longer valid. A remarkable feature of the relation (44) is its independence of the coupling parameters t and J .

For the two-leg ladder the modified d-wave relation (44) turns into

$$\Delta_{\perp} = -2\Delta_{\parallel}. \quad (55)$$

This result agrees with the available numerical results [2,3]. It would provide a good test for possible numerical models on ladders with a higher number of legs.

The d-wave relation (44) may be easily extended to a non-isotropic coupling. In fact, the whole slave boson mean-field theory may be rederived for arbitrary coupling constants t_{ij} and J_{ij} , differing at different links. One just needs to replace t and J in Eqs. (7) – (10) by t_{ij} and J_{ij} . The whole argument of this section may be repeated down to Eq.(53) which we must now replace by the Hamiltonian

$$H_0 = \frac{1}{2} \sum_{\{ij\}} t_{ij} (b_{1i}^{\dagger} \chi_{ij} b_{1j} + b_{1j}^{\dagger} \chi_{ij}^* b_{1i}). \quad (56)$$

Finally, this leads to the following generalization of the d-wave relation (44):

$$\sum_j t_{ij} \Delta_{ij}^{(0)} = 0. \quad (57)$$

This equation implies that as the hopping on a link increases, the weight of the superconducting order parameter on this link decreases. Of course, Eq.(57) may also be derived exactly for the Hubbard model (with the site- and link-dependent U and t) by the method of [12].

V. CONCLUSION

We presented the low-energy effective theory for charge excitations in two-leg t-J ladder, based on the mean-field treatment of spin degrees of freedom. We found that the $SU(2)$ slave-boson formalism predicts bipolaronic picture of charge excitations, as expected from earlier analytic and numerical works. While capturing well the low-energy physics, our approximation is not reliable for spin and single-electron excitations which have a gap of order J . In the framework of the model developed in the paper, single-hole excitations may be constructed as holon-spinon pairs bound by confining gauge-field interaction. The similarity of the boson-fermion spectrum in Fig. 7 to the single-hole spectrum found numerically in [10] makes this possibility very appealing. However, spin structure of the ladder was included in the lowest mean-field order, and to restore correctly spin excitations requires a more elaborate treatment.

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